Existence and Uniqueness of the Market Equilibrium in a VES Economy

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We adopt the VES demand structure of Dixit and Stiglitz and the heterogeneous firm framework of Melitz, and provide sufficient conditions for a market equilibrium in this setting.

Consumers

A mass L of identical consumers in an economy are each endowed with one unit of labor and face a wage rate w normalized to one. Preferences are identical across all consumers. Let M_e denote the mass of entering varieties and $q(c)$ denote the quantity consumed of variety c by each consumer, where there are a continuum of varieties $[0, c_d]$ distributed with cdf G with non-vanishing pdf in a neighborhood of zero.[1] A consumer has preferences over differentiated goods $U(M_e, c_d, q)$ which take the general VES form:

$$U(M_e, c_d, q) \equiv M_e \int_0^{c_d} u(q(c))\, dG(c). \tag{1}$$

Consumers face a price for each variety c of $p(c)$ and thus a budget constraint of

$$M_e \int_0^{c_d} p(c)\, q(c)\, dG(c) = 1.$$

Without loss of generality, assume $p(c)$ is strictly increasing in c. We will assume a set of regularity conditions to ensure the existence of a market equilibrium. For the consumer's budgeting problem to be well defined, we make Assumptions 1 and 2.

[1] As is typical, all primitive model functions are assumed to be four times continuously differentiable.

Assumption 1. *Utility Restrictions.*

1. *(Regularity) u is strictly increasing, concave, satisfies inada conditions and $u(0)=0$.*[2]

2. *(Bounded Elasticities) The elasticity of utility $\varepsilon(q)$ and elasticity of marginal utility $\mu(q)$ are bounded below by $m>0$ and above by $1-m<1$.*

While Assumption 1.1 is fairly standard, 1.2 maintains boundedness between aggregate costs, revenues and welfare. The following Assumption 2 will be ensured in equilibrium once firm behavior is considered.

Assumption 2. *Consumer Regularity Conditions.*

1. *(Non-satiation)* $\sup_q \left\{U(M_e,c_d,q):\ M_e\int_0^{c_d} p(c)\,q(c)\,dG(c)=1\right\}<\infty$.

2. *(Bounded Expenditure)* $\int_0^{c_d} p(c)\,(u')^{-1}\left(\delta^{\text{finite}}p(c)\right)dG(c)<\infty$ for some $\delta^{\text{finite}}>0$.[3]

Assumption 2.1 is automatically satisfied if u is bounded, but more broadly is an assumption that the prices faced by a consumer do not allow consumers to attain infinite welfare conditional on the distribution of prices, for instance if many goods have prices close to zero. Assumption 2.2 is a condition that guarantees the prices presented to consumers imply finite expenditure.

Under Assumptions 1 and 2, the form and boundedness of the maximization problem and strict concavity of $u(q)-\delta\cdot p(c)\,q$ in q for any $\delta\in(0,\infty)$ implies that if there exists a consumer budget multiplier $\delta^{\text{cons}}\in(0,\infty)$ such that the first order condition and constraint

$$u'\left(q^{\text{cons}}(c)\right)=\delta^{\text{cons}}p(c)\quad\text{and}\quad M_e\int_0^{c_d} p(c)\,q^{\text{cons}}(c)\,dG(c)=1 \tag{2}$$

are satisfied, then Equation (2) determines the unique solution $q^{\text{cons}}(c)=(u')^{-1}\left(\delta^{\text{cons}}p(c)\right)$ to the consumer's budget problem.[4]

Lemma 1. *Under Assumption 2, there exists a unique solution to the consumer problem.*

Proof. Necessarily any such δ^{cons} must satisfy $\Delta^{\text{cons}}\left(\delta^{\text{cons}}\right)=1$ where

$$\Delta^{\text{cons}}(\delta)\equiv M_e\int_0^{c_d} p(c)\left(u'\right)^{-1}\left(\delta p(c)\right)dG(c). \tag{3}$$

[2] Utility functions not satisfying inada conditions are permissible but may require parametric restrictions to ensure existence.

[3] Note that regularity conditions on G are required for existence of a market equilibrium in Melitz (2003) as would be implied by this assumption.

[4] $\delta^{\text{cons}}\in(0,\infty)$ and the assumptions on u guarantee this q^* is well defined, finite and positive.

As the integrand of the RHS of Equation (3) is strictly decreasing in δ, Δ^{cons} is strictly decreasing in δ. Inada conditions on u along with monotone convergence also implies that $\lim_{\delta\to 0}\Delta^{\text{cons}}(\delta)=\infty$ and with Assumption 2.2 implies $\lim_{\delta\to\infty}\Delta^{\text{cons}}(\delta)=0$ by monotone convergence of $\Delta^{\text{cons}}\left(\delta^{\text{finite}}\right)-\Delta^{\text{cons}}(\delta)$. This guarantees the uniqueness and existence of δ^{cons} and thus optimal consumption through Equation (2). □

Firms

There is a continuum of firms which may enter the market for differentiated goods by paying a sunk entry cost f_e. Each firm produces a unique variety, so the mass of entering firms is the mass of entering varieties M_e. Upon entry, each firm receives a unit cost $c\geq 0$ drawn from the distribution G. After entry, should a firm produce, it incurs a fixed cost of production f. For each variety c, VES preferences induce an inverse demand $p(q(c))=u'(q(c))/\delta$ where δ is a consumer's budget multiplier. Each firm acts as a monopolist of variety c facing a mass L of consumers, with profits

$$\pi(c)\equiv\max_{q(c)}\left[p(q(c))-c\right]q(c)L-f=\max_{q(c)}\left[u'(q(c))\,q(c)/\delta-cq(c)\right]L-f.$$

Free entry implies that ex ante average profits must equal sunk entry costs, so $\int\pi(c)\,dG(c)=f_e$.

To ensure that firm profit maximization is well behaved, we make Assumption 3.

Assumption 3. *Firm Regularity Conditions.*

1. *(Decreasing Marginal Revenue) Real revenues $u'(q)\cdot q$ are strictly concave in quantity.*[5]
2. *(Bounded Costs)* $\int_0^{c_d} c\cdot(u')^{-1}\left(\delta^{\text{finite}}c\right)dG(c)<\infty$ for some $\delta^{\text{finite}}>0$.

Assumption 3.1 guarantees the monopolist's FOC is optimal, the quantity choice is determined by the equality of marginal revenue and marginal cost, and that quantities are uniquely defined for any positive, finite δ. Assumption 3.2 is a condition that guarantees the distribution of costs in conjunction with demand allows for finite resource usage by a unit mass of firms. Furthermore, inada conditions on $u'(q)\cdot q$ imply

$$\lim_{c\to 0}q(c)=\infty,\qquad\lim_{c\to\infty}q(c)=0,\qquad\lim_{\delta\to 0}q(c)=\infty\quad\text{and}\quad\lim_{\delta\to\infty}q(c)=0.\tag{4}$$

[5]Inada conditions for revenue are implied by Assumption 1 since $[u'(q)\cdot q]'=u'(q)[1-\mu(q)]$.

Specifically, the markup rate is $(p(c)-c)/p(c) = |u''(q)\cdot q/u'(q)| = \mu(q(c))$. Therefore profits may be written $\pi(c) = \mu(q(c))/[1-\mu(q(c))]\cdot Lcq(c) - f$, so Assumption 1.2 implies

$$\pi(c) \in [m/(1-m)\cdot Lcq(c) - f, (1-m)/m\cdot Lcq(c) - f]. \tag{5}$$

Equation (5) with Equation (4) implies

$$\lim_{c\to 0}\pi(c) = \infty, \qquad \lim_{c\to\infty}\pi(c) = -f, \qquad \lim_{\delta\to 0}\pi(c) = \infty \quad \text{and} \quad \lim_{\delta\to\infty}\pi(c) = -f. \tag{6}$$

Since $\pi(c)$ is strictly decreasing in c and δ, clearly from Equation (4) there is a unique cost cutoff c_d for every δ where $\pi(c_d) = 0$.

Lemma 2. *Under Assumption 3 there is a unique δ^* for which firms' entry, production and quantity decisions are optimal. Furthermore, Assumption 2.2 holds at δ^*.*

Proof. By Assumption 3.2, there is some δ^{finite} where, for $\delta = \delta^{\text{finite}}\cdot(1-m)$,

$$\begin{aligned}\int_0^{c_d}[\pi(c)+f]\,dG(c) &\le \frac{1-m}{m}L\int_0^{c_d} cq(c)\,dG(c)\\ &= \frac{1-m}{m}L\int_0^{c_d} c\left(u'\right)^{-1}\left(\frac{\delta c}{1-\mu(q(c))}\right)dG(c)\\ &\le \frac{1-m}{m}L\int_0^{c_d} c\left(u'\right)^{-1}\left(\delta^{\text{finite}}c\right)dG(c),\end{aligned}$$

and therefore average profits $\int_0^{c_d}\pi(c)\,dG(c)$ and average costs $\int_0^{c_d} Lcq(c)\,dG(c)$ are bounded at $\delta = \delta^{\text{finite}}\cdot(1-m)$.

Now note that c_d, $q(c)$ and $\pi(c)$ are strictly decreasing in δ so that average profits and costs are strictly decreasing in δ. It follows from Equation (4) and dominated convergence that for some open interval $(\underline{\delta},\infty)$ where average profits are bounded and positive that

$$\lim_{\delta\to\underline{\delta}}\int_0^{c_d}\pi(c)\,dG(c) = \infty, \qquad \lim_{\delta\to\infty}\int_0^{c_d}\pi(c)\,dG(c) = -f\cdot\lim_{\delta\to\infty}G(c_d) = 0$$

so that there is a unique δ^* s.t. $\int_0^{c_d}\pi(c)\,dG(c) = f_e$. Thus firms' entry, production and quantity decisions are optimal at δ^*.

Finally, from Assumption 1.2 and bounding $\mu/(1-\mu)$ by $m/(1-m)$ and $(1-m)/m$, it is clear that average costs are also bounded and positive at δ^*. It follows that average revenues $\int_0^{c_d} L[u'(q(c))/\delta]\,q(c)\,dG(c)$ (equal to average profits plus average costs) are bounded and positive. This shows exactly that Assumption 2.2 holds at δ^*. □

So far we have specified a particular $\delta = \delta^*$, and thus candidates for prices and a range of firm types $[0, c_d]$ who produce varieties for consumers. A natural candidate for the mass of entrants M_e is determined by the resource constraint of the economy at δ^* by

$$M_e \equiv L / \left[\int_0^{c_d} [Lcq(c) + f] \, dG(c) + f_e \right].$$

If this unique specification for firm behavior satisfies Assumption 2.1, then as we have shown above, it is the unique equilibrium of the economy. The key argument in this respect is that the bounded costs Assumption 3.2 implies bounded revenues and thus bounded utility through Assumption 1.2.

Lemma 3. *Under Assumptions 1 and 3 there is a unique market equilibrium.*

Proof. As discussed above, what is required is that Assumption 2.1 holds. At $(M_e, c_d, p(c))$ uniquely fixed by δ^* and $q^*(c) \equiv (u')^{-1}(\delta^* p(c))$, consider the strictly concave problem

$$\max_q M_e \int_0^{c_d} u(q(c)) \, dG(c) \text{ subject to } M_e \int_0^{c_d} p(c) q(c) \, dG(c) = 1.$$

Letting δ denote the Lagrange multiplier for this problem, the FOC is $u'(q(c)) = \delta p(c) = \delta u'(q^*(c)) / \delta^*$, which clearly holds at $q = q^*$ and $\delta = \delta^*$, while by construction of δ^* the constraint exactly holds. Provided $M_e \int_0^{c_d} u(q^*(c)) \, dG(c)$ is finite then it must equal $\sup_q \left\{ U(M_e, c_d, q) : \, M_e \int_0^{c_d} p(c) q(c) \, dG(c) = 1 \right\}$ by sufficiency of the FOC. Entry M_e is finite and positive by construction, and consider that

$$\begin{aligned}
\int_0^{c_d} u(q^*(c)) \, dG(c) &= \int_0^{c_d} \left[u'(q^*(c)) \, q^*(c) / \varepsilon(q(c)) \right] dG(c) \\
&\leq (1/m) \cdot \int_0^{c_d} u'(q^*(c)) \, q^*(c) \, dG(c) \\
&= (1/m) \, \delta^* / M_e < \infty
\end{aligned}$$

which shows that Assumption 2.1 holds. □

References

Melitz, Marc J., "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 2003, *71* (6), 1695–1725.